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Symmetry classification and exact solutions of the one-dimensional Fokker–Planck equation with arbitrary coefficients of drift and diffusion

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Abstract. Symmetry properties of the one-dimensional Fokker–Planck equations with arbitrary coefficients of drift and diffusion are investigated. It is proved that the group symmetry of these equations can be one, two, four or six parametric and corresponding criteria are obtained. The changes of the variables reducing Fokker–Planck equations to the heat equation and Schrödinger equation with certain potentials are determined. Using the substructure of the invariance algebra of the Rayleigh process equation we obtain the differential invariants and construct some classes of the exact solutions of this equation.

1. Introduction

The Fokker–Planck equation (FPE) is the basic equation in the theory of continuous Markovian processes. In a one-dimensional case the FPE has the form [1, 2]

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}[A(t,x)u] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(t,x)u] \tag{1}$$

where $u = u(t,x)$ is the probability density, and $A(t,x)$ and $B(t,x)$ are differentiable functions: the coefficients of drift and diffusion, respectively.

The FPE serves as a mathematical model for a number of problems in the physical and biological sciences (see [1–10]). Blumen and Cole [11] used the Lie symmetry method to find the invariant solutions of the heat equation and Bluman did the same thing for a special case of equation (1). He also showed that every one-dimensional FPE with a six-parameter group of Lie symmetry can be transformed to a diffusion heat equation [12]. Recently, Sastry and Dunn [13], Cicogna and Vitali [14] and Shtelen and Stognii [15] applied Lie's extended group method to investigate the symmetry structure of some interesting cases of the Fokker–Planck-type equations.

We have investigated symmetry properties of equation (1) under the infinitesimal basis operators [16–18]

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2}$$

The symmetry operators are defined from an invariance condition

$$\underset{2}{\hat{X}}\, L|_{L=0} = 0 \tag{3}$$

where

$$L = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[A(t,x)u] - \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(t,x)u].$$

$\hat{X}_{2}$ is the second prolongation of the operator X, which is constructed according to the formulae [16–18]:

$$\begin{aligned}
\hat{X}_{2} &= X + \theta_t \frac{\partial}{\partial u_t} + \theta_x \frac{\partial}{\partial u_x} + \theta_{xx}\frac{\partial}{\partial u_{xx}} \\
\theta_t &= D_t\eta - u_t D_t\xi^0 - u_x D_t \xi^1 \\
\theta_x &= D_x\eta - u_t D_x\xi^0 - u_x D_x \xi^1
\end{aligned}$$

where

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} \\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x}.
\end{aligned}$$

From these formulae and the condition of invariance (3), equating coefficients by a function u and its derivatives $u_x, u_{tt}, u_{tx}, u_{xx}$ (u_t can be expressed from equation (1)) to zero it is possible to determine the following system of equations on functions ξ^0, ξ^1, η:

$$\begin{aligned}
&\xi^0 = \xi^0(t) \qquad \xi^1 = \xi^1(t,x) \qquad \eta = \chi(t,x)u \\
&2\xi^1_x B - \xi^0_t B - \xi^1 B_x - \xi^0 B_t = 0 \\
&\xi^0_t(A - B_x)\xi^1_t + \xi^0(A_t - B_{tx}) + \xi^1(A_x - B_{xx}) - \xi^1_x(A - B_x) + \tfrac{1}{2}B\xi^1_{xx} = B\chi_x \\
&\chi_t + \xi^0_t(A_x - \tfrac{1}{2}B_{xx}) + \xi^0(A_{tx} - \tfrac{1}{2}B_{txx}) + \xi^1(A_{xx} - \tfrac{1}{2}B_{xxx}) + \chi_x(A - B_x) - \tfrac{1}{2}B\chi_{xx} = 0
\end{aligned} \tag{4}$$

where subscripts t, x mean differentiation on appropriate variables. Let us also introduce the following notation: $\frac{\partial}{\partial t} = \partial_t$, $\frac{\partial}{\partial x} = \partial_x$, $\frac{\partial}{\partial u} = \partial_u$.

Remark. The symmetry operators $f(t,x)\partial_u$, where $f(t,x)$ is any solution of equation (1), are excluded from further consideration.

In section 2 the necessary and sufficient condition under which the FPE is reduced to a homogeneous FPE is obtained. In section 3 we find the criterion of invariance of the FPE under four- and six-parametrical groups and the corresponding formulae transforming either to heat or Schrödinger equations are calculated in section 4. Finally, in section 5 we give several important examples of homogeneous FPEs and construct some classes of exact solutions for the equation describing the Rayleigh process.

2. The basic theorem

Theorem 1. *If a symmetry operator (2) $Q \neq u\partial_u$ for FPE (1) exists then we have the following kind of transformation:*

$$\tilde{t} = T(t) \qquad \tilde{x} = X(t,x) \qquad u = v(t,x)\tilde{u}$$

which reduces it to equation (1) with coefficients of drift and diffusion $\tilde{A} = A(\tilde{x})$, $\tilde{B} = B(\tilde{x})$.

Proof. A condition of the theorem implies that the solution ξ^0, ξ^1, χ of the determining equations (4) exists such that either ξ^0 or ξ^1 are not identically equal to zero. Actually, from conditions $\xi^0 \equiv \xi^1 \equiv 0$ it follows that $\chi = \lambda = \text{const}$, i.e. $Q = u\partial_u$. Furthermore, two alternatives will be considered:

(1) there is a solution $\xi^0 \not\equiv 0, \xi^1, \chi$;
(2) there is a solution $\xi^0 \equiv 0, \xi^1 \not\equiv 0, \chi$ (in this case all $\xi^0 \equiv 0$).

We shall prove the theorem for each of these alternatives.

(1) Let $\xi^0 \not\equiv 0, \xi^1, \chi$ be a solution of the determining equations (4). We consider transformations

$$\tilde{t} = T(t) \qquad \tilde{x} = \omega \qquad u = v(t, x)\tilde{u} \tag{5}$$

where $T(t) = \int \frac{\mathrm{d}t}{\xi^0(t)}$, and the functions $\omega = \omega(t, x)$, $v(t, x)$ satisfy the equations

$$\begin{aligned} &\xi^0\omega_t + \xi^1\omega_x = 0 \\ &\xi^0 v_t + \xi^1 v_x = \chi v \end{aligned} \tag{6}$$

where $\omega \neq$ const is any fixed solution of equation (6). It is easy to show that the symmetry operator

$$Q = \xi^0\partial_t + \xi^1\partial_x + \chi u\partial_u$$

with the new variables $(\tilde{t}, \tilde{x}, \tilde{u})$ has the form $\tilde{Q} = \partial_{\tilde{t}}$. Let us show that transformation (5) exists which reduces equation (1) to a FPE with coefficients $\tilde{A}(\tilde{t}, \tilde{x})$, $\tilde{B}(\tilde{t}, \tilde{x})$. However, as the transformed equation has the symmetry operator $\tilde{Q} = \partial_{\tilde{t}}$, it follows that $\tilde{A} = \tilde{A}(\tilde{x})$, $\tilde{B} = \tilde{B}(\tilde{x})$.

So, by applying transformation (5) to equation (1) we find the following equation:

$$\begin{aligned} \tilde{u}_{\tilde{t}} = \frac{\xi^0}{v}\{&[-v_t + (\tfrac{1}{2}B_{xx} - A_x)v + (B_x - A)v_x + \tfrac{1}{2}Bv_{xx}]\tilde{u} \\ &+[-v\omega_t + (B_x - A)v\omega_x + \tfrac{1}{2}B(2v_x\omega_x + v\omega_{xx})]\tilde{u}_{\tilde{x}} + \tfrac{1}{2}Bv\omega_x^2\tilde{u}_{\tilde{x}\tilde{x}}\}. \end{aligned} \tag{7}$$

This also means that in expressions dependent on variables (t, x) it is necessary to make the replacement $(t, x) \to (\tilde{t}, \tilde{x})$. For equation (7) to be a FPE it is necessary that the unknown coefficients $\tilde{A}(\tilde{t}, \tilde{x})$, $\tilde{B}(\tilde{t}, \tilde{x})$ satisfy the following equations:

$$\begin{aligned} &\tilde{B}(\tilde{t}, \tilde{x}) = B\xi^0\omega_x^2 \\ &\tilde{B}_{\tilde{x}} - \tilde{A} = -\xi_0\omega_t + (B_x - A)\xi_0\omega_x + \frac{v_x}{v}\xi_0\omega_x B + \frac{1}{2}\xi_0\omega_{xx}B \\ &\frac{1}{2}\tilde{B}_{\tilde{x}\tilde{x}} - \tilde{A}_{\tilde{x}} = \xi_0\frac{v_t}{v} + \left(\frac{1}{2}B_{xx} - A_x\right)\xi_0 + \xi_0\frac{v_x}{v}(B_x - A) + \frac{1}{2}\frac{v_{xx}}{v}\xi_0 B. \end{aligned} \tag{8}$$

(a) We now consider the first of equations (8). Let us show that $\partial_{\tilde{t}}\tilde{B} = 0$. From transformations (5), (6) one may find that

$$\partial_{\tilde{t}} = \xi^0\partial_t + \xi^1\partial_x. \tag{9}$$

Then, from the first equations of (6) and (9) it follows that

$$\partial_{\tilde{t}}(B\xi^0\omega_x^2) = \xi^0\omega_x^2[\xi^0 B_t + \xi^1 B_x + \xi_t^0 B - 2B\xi_x^1] = 0.$$

The last equality is carried out by virtue of equations (4).

(b) Now we consider the second equation of system (8). From the first equation of this system we have

$$\tilde{B}_{\tilde{x}} = \frac{\xi^0}{\omega_x}\partial_x(B\omega_x^2).$$

Substituting this expression into the second equation of system (8) we find $\tilde{A}$:

$$\tilde{A} = \xi^0\omega_t + A\xi^0\omega_x - \frac{v_x}{v}\xi^0\omega_x B + \frac{3}{2}\xi^0\omega_{xx}B. \tag{10}$$

As well as in item (a) it is possible to show that the consequence of the system of equations (4) is

$$\partial_{\tilde{t}}\tilde{A} = (\xi^0\partial_t + \xi^1\partial_x)\left[\xi^0\omega_t + A\xi^0\omega_x - \frac{v_x}{v}\xi^0\omega_x B + \frac{3}{2}\xi^0\omega_{xx}B\right] = 0.$$

In the following $\tilde{A} = \tilde{A}(\tilde{x})$.

(c) Let us consider the third equation of system (8). The left-hand side of the equation $\frac{1}{2}\tilde{B}_{\tilde{x}\tilde{x}} - \tilde{A}_{\tilde{x}} = F(\tilde{x}) = F(\omega)$ follows from items (a) and (b). The general solution of the second equation (6) is $v(t,x) = v^*(t,x)G(\omega)$, where $v^*(t,x)$ is some solution of this equation, and $G(\omega)$ is an arbitrary function. Substituting this relation into the right-hand side of the third equation in (8) we obtain

$$F(\omega) = F_1(t,x) + F_2(t,x)G' + F_3(t,x)[G'' + G'^2] \qquad F_3(t,x) \neq 0.$$

By analogy with the items (a), (b) taking into account (4), (6), we come to

$$\partial_{\tilde{t}}F_i(t,x) = [\xi^0\partial_t + \xi^1\partial_x]F_i(t,x) = 0 \qquad i = 1, 2, 3.$$

Then $F_i = F_i(\tilde{x}) = F_i(\omega)$. Finally, we have

$$F(\omega) = F_1(\omega) + F_2(\omega)G' + F_3(\omega)[G'' + G'^2] \qquad F_3 \neq 0.$$

Choosing function $G(\omega)$ as some solution of this equation, we find transformation (5), where $T = \int \frac{\mathrm{d}t}{\xi^0}$, $v(t,x) = v^*(t,x)G(\omega)$, which reduces the appropriate FPE (1) to a FPE with coefficients $\tilde{B}(\tilde{x})$, $\tilde{A}(\tilde{x})$.

(2) Let there be a solution of the determining equations (4) such that $\xi^0 \equiv 0, \xi^1 \neq 0, \chi$. In this case we choose transformations

$$\tilde{t} = t \qquad \tilde{x} = R(t,x) \qquad u = v(t,x)\tilde{u} \tag{11}$$

under the condition of

$$\xi^1 R_x = 1 \qquad \xi^1 v_x = \chi v. \tag{12}$$

The proof of the existence of transformations (11), (12) which to do not change the form of FPE is similar to the proof given in item (1). Under these the transformation operator $Q = \xi^1\partial_x + \chi u\partial_u$ is reduced to $\tilde{Q} = \partial_{\tilde{x}}$. Then the new coefficients $\tilde{A}$, $\tilde{B}$ in the transformed FPE depend only on t. As is known, such equations are reduced to the heat equation by transformations of the following kind:

$$\tilde{t} = T(t) \qquad \tilde{x} = R(t,x) \qquad u = v(t,x)\tilde{u}. \tag{13}$$

The theorem is, therefore, proved. □

Theorem 2. *The dimension of an invariance algebra of FPE (1) can be equal to 1, 2, 4 or 6.*

Proof. If the dimension of algebra is greater than 1 then equation (1) is reduced to the equation with $\tilde{A} = \tilde{A}(\tilde{x})$, $\tilde{B} = \tilde{B}(\tilde{x})$. Classification of such equations is also known: the dimension of their invariance algebra is either 2, 4 or 6 [14]. □

3. Criterion of invariance FPE under four- and six-parametrical groups of symmetry

In [19] it is shown that any diffusion process with coefficients of drift $A(t, x)$ and diffusion $B(t, x)$ can be reduced to a process with appropriate coefficients $\tilde{A}(t, x) = A(t, x)/B(t, x)$ and $\tilde{B}(t, x) = 1$ through the random replacement of time $\tau(t)$. Using the result of theorem 1 we perform a symmetry classification of FPE for the coefficients $B(t, x) = 1$ and any $A(t, x)$ just as was done in [14] for the case $A = A(x)$ (a homogeneous process). So, by putting $B = 1$ into equations (4) it is easy to show that

$$\begin{gathered} \xi^0 = \tau(t) \qquad \xi^1 = \tfrac{1}{2}x\tau' + \varphi(t) \\ \tfrac{3}{2}\tau' M + \tau M_t + (\tfrac{1}{2}\tau' x + \varphi)M_x = \tfrac{1}{2}\tau' x + \varphi'' \\ \chi = \tfrac{1}{2}\tau' x A(t, x) - \tfrac{1}{4}x^2\tau'' - \varphi' x + \varphi A(t, x) + \tau \int_{x_0}^{x} \frac{\partial A(t, \xi)}{\partial t}\, \mathrm{d}\xi + \theta(t) \end{gathered} \tag{14}$$

where $M = A_t + \frac{1}{2}A_{xx} + AA_x$, x_0 and $\theta(t)$ are arbitrary point and function, respectively. Let us find a condition on M under which there exists at least two linearly independent solutions $\tau(t)$ of equations (14). In this case, from theorem 2, it follows that there exists either three or five operators of symmetry (besides trivial $u\partial_u$). Let us assume that $M_{xx} \neq 0$. After differentiating both parts (14) twice on x we have

$$\tfrac{5}{2}\tau' M_{xx} + \tau M_{txx} + (\tfrac{1}{2}\tau' x + \varphi)M_{xxx} = 0. \tag{15}$$

Now, if we assume that $M_{xxx} = 0$, i.e. $M_{xx} = F(t)$, then the following condition holds:

$$\tfrac{5}{2}\tau' F + \tau F' = 0. \tag{16}$$

For this equation there is only one linearly independent solution, therefore $M_{xxx} \neq 0$. Then from (15):

$$-\varphi(t) = \frac{5M_{xx} + xM_{xxx}}{2M_{xxx}}\tau' + \frac{M_{txx}}{M_{xxx}}\tau = h(t, x)\tau' + r(t, x)\tau.$$

So if (τ_1, φ_1), (τ_2, φ_2) are linearly independent then τ_1, τ_2 are linearly independent, and also $h_x\tau' + r_x\tau = 0$. Thus,

$$h_x\tau_1' + r_x\tau_1 = 0 \qquad h_x\tau_1' + r_x\tau_1 = 0.$$

As the Wronskian $\begin{vmatrix} \tau_1' & \tau_1 \\ \tau_2' & \tau_2 \end{vmatrix} \neq 0$, then from this system it follows that $h_x \equiv 0$, $r_x \equiv 0$, i.e.

$$\frac{5M_{xx} + xM_{xxx}}{2M_{xxx}} = h(t) \qquad \frac{M_{xxt}}{M_{xxx}} = r(t). \tag{17}$$

From conditions (17) it is easy to deduce that

$$M = \lambda(x - H(t))^{-3} + F(t)x + G(t) \tag{18}$$

where $\lambda = \text{const} \neq 0$, and H, F, G are arbitrary functions. Now notice that if $M_{xx} = 0$, M has the form of (18) with $\lambda = 0$. Thus condition (18) is necessary for the invariance algebra to possess dimension four or six. Substituting (18) into (15) and equating zero factors at $x - H$, $(x - H)^{-4}$ and 1, we obtain the following conditions:

$$\begin{gathered} 2\tau' F + \tau F' = \tfrac{1}{2}\tau''' \qquad \lambda(\tau H' - \tfrac{1}{2}\tau' H - \varphi) = 0 \\ \tfrac{3}{2}\tau'(FH + G) + \tau(F'H + G') + F(\tfrac{1}{2}\tau' H + \varphi) = \tfrac{1}{2}\tau''' H + \varphi'''. \end{gathered} \tag{19}$$

(1) Let $\lambda \neq 0$. Deriving $\varphi(t) = \tau H' - \frac{1}{2}\tau' H$ from the second equation and substituting it into the third equation we have

$$\tfrac{3}{2}\tau'(FH + G - H'') + \tau(FH + G - H'')' = 0.$$

Due to the condition of existence at least two independent solutions τ_1, τ_2 result in the equation $FH + G - H'' = 0$. In this case the number of fundamental solutions of system (19) is three. Actually, there are three linear independent solutions τ_1, τ_2, τ_3 of the first equation of (19). From the second equation of (19) φ_i is expressed by τ_i, $i = 1, 2, 3$.

(2) If $\lambda = 0$, the system of equations (19) has five linearly independent solutions (τ_i, φ_i), $i = \overline{1, 5}$.

The theorem is therefore proved.

Theorem 3. *(1) The class FPE (1) with $B = 1$ admitting four-dimensional algebra of invariance is described by the condition*

$$A_t + \tfrac{1}{2}A_{xx} + AA_x = \lambda(x - H(t))^{-3} + F(t)x + G(t) \tag{20}$$

where $\lambda = const \neq 0$, G satisfies the condition

$$G = H'' - FH \tag{21}$$

where $F(t)$, $H(t)$ are arbitrary functions.

(2) The class FPE (1) with $B = 1$ admitting six-dimensional algebra of invariance is described by condition (20) in which $\lambda = 0$, F, G are arbitrary functions.

Remark. In particular, if the coefficient $A(t, x)$ satisfies the Burgers equation then FPE (1) is reduced to the heat equation (see [20]).

4. Transformation of the Fokker–Planck equations to homogeneous equations

(1) It turns out that FPE (1) ($B = 1$), (20) at $\lambda = 0$ is reduced to the heat equation [20]. We find the appropriate transformation (5), (6). Let τ be any solution of system (19) and $\tau > 0$ (evidently it is always possible to choose the solution $\tau(t) > 0$ on some interval). From the formulae (6), (14) it is easy to prove that $\omega(t, x) = \tau^{1/2}x - \int_{t_0}^{t} \varphi(\xi)\tau^{-3/2}(\xi)\,\mathrm{d}t$, where t_0 is an arbitrary fixed point. Let us consider the transformation

$$\begin{aligned} \tilde{t} &= \frac{1}{2}\int \frac{\mathrm{d}t}{\tau} \\ \tilde{x} &= \omega(t, x) = \tau^{-1/2}x - \int_{t_0}^{t} \varphi(\xi)\tau^{-3/2}(\xi)\,\mathrm{d}\xi \\ u(t, x) &= v(t, x)\tilde{u}(\tilde{t}, \tilde{x}). \end{aligned} \tag{22}$$

Having made in (1), (20) the replacement variable (22) we arrive at the equation

$$\begin{aligned} \tilde{u}_{\tilde{t}} = -2\tau\left(\frac{v_t}{v} + A_x + A\frac{v_x}{v} - \frac{1}{2}\frac{v_{xx}}{v}\right)\tilde{u} \\ -2\left(-\frac{1}{2}\tau^{1/2}\tau' x - \varphi\tau^{-1/2} + A\tau^{1/2} - \frac{v_x}{v}\tau^{1/2}\right)\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}}. \end{aligned} \tag{23}$$

Equating the factor at $\tilde{u}_{\tilde{x}}$ to zero, we obtain

$$v = \exp\left(-\tfrac{1}{4}\tau^{-1}\tau' x^2 - \tau^1\varphi x + \int_{x_0}^{x} A(t, \xi)\,\mathrm{d}\xi + h(t)\right) \tag{24}$$

where $h(t)$ is an arbitrary function and x_0 is some fixed point. Substituting (24) into the expression $\frac{v_t}{v} + A_x + A\frac{v_x}{v} - \frac{1}{2}\frac{v_{xx}}{v}$ (factor at $\tilde{u}$ in (23)) and equating it to zero we get

$$h'(t) = \tfrac{1}{2}[\tau^{-2}\varphi^2 - \tfrac{1}{2}\tau^{-1}\tau^1 - A_x(t, x_0) - A^2(t, x_0)] \tag{25}$$

$$\tfrac{1}{2}\tau^{-1}\tau'' - \tfrac{1}{4}\tau^2(\tau')^2 = F \qquad \tau^{-1}\varphi' - \tfrac{1}{2}\tau^2\tau'\varphi = G. \tag{26}$$

It is easy to prove that if ($\tau \neq 0, \varphi$) is some solution of system (26) then it satisfies system (19) ($\lambda = 0, M = 0$). Then we have transformation (22), where functions $v(t, x)$, $\tau(t)$, $\varphi(t)$ can be found from (24)–(26), reducing FPE (1), (20) ($\lambda = 0$) to the heat equation

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}}. \tag{27}$$

Note that system (26) is reduced to the following:

$$2y' + y^2 = 4F \qquad y = \frac{\tau'}{\tau} \qquad \varphi = \tau^{1/2}\int_{t_0}^{t} \tau^{1/2} G\,\mathrm{d}t. \tag{28}$$

(2) We now consider FPE (1), (20) with $\lambda \neq 0$. As in case (1), transformation (22) reduces this equation to (23). The conditions for (23) to be a FPE are as follows:

$$\begin{aligned} \tilde{A} &= \tilde{A}(\omega) = -\tau^{-1/2}\tau' x - 2\varphi\tau^{-\frac{1}{2}} + 2A\tau^{1/2} - 2\tau^{1/2}\frac{v_x}{v} \\ \tilde{A}_\omega &= 2\tau\left(\frac{v_t}{v} + A_x + A\frac{v_x}{v} - \frac{1}{2}\frac{v_{xx}}{v}\right) \end{aligned} \tag{29}$$

where ω is given in (22). The first condition is equivalent to the equation (see (9))

$$\partial_{\tilde{t}}\tilde{A} = \left[\tau\partial_t + \left(\frac{1}{2}\tau' x + \varphi\right)\partial_x\right]\left(-\tau^{-1/2}\tau' x - 2\varphi\tau^{\frac{1}{2}} + 2A\tau^{1/2} - 2\tau^{1/2}\frac{v_x}{v}\right) = 0. \tag{30}$$

Omitting intermediate calculations we give the general solution $v(t, x)$ of equation (30):

$$v(t, x) = \exp\left[\int_{x_0}^{x} A(t, \xi)\,\mathrm{d}\xi - \tfrac{1}{4}\tau^{-1}\tau' x^2 - \tau^{-1}\varphi x + k(\omega)\right] \tag{31}$$

where $k(\omega)$ is an arbitrary function, x_0 is some fixed point. Substituting (31) into the first equation of (29) one can easily verify that $\tilde{A} = -k'(\omega)$ ($k'(\omega) = \frac{\mathrm{d}k(\omega)}{\mathrm{d}\omega}$). Furthermore, let us substitute $\tilde{A}(\omega) = -k'(\omega)$, $v(t, x)$ (31) into the second equation of (29). Having chosen the conditions

$$\tau^{1/2}\int_{t_0}^{t}\varphi\tau^{-3/2}\,\mathrm{d}t = H \qquad \tfrac{1}{2}\tau^{-1}\tau'' - \tfrac{1}{4}\tau^{-2}\tau'^2 = F \qquad \tau^{-1}\varphi' - \tfrac{1}{2}\tau^{-2}\tau'\varphi = G \tag{32}$$

$$k'' - k'^2 = \lambda\omega^{-2} \tag{33}$$

the second equation of (29) is satisfied. Condition (32) is a possible choice because, as it is easy to verify, any solution $\tau \neq 0, \varphi$ of the given system is a particular solution of the equation systems (19), (21). System (32), taking into account (21), is equivalent to

$$2y' + y^2 = 4F \qquad y = \frac{\tau'}{\tau} \qquad \varphi = \tau^{3/2}(\tau^{-1/2}H)'. \tag{34}$$

Thus the theorem is proved.

Theorem 4. *FPE (1), (20), (21) with $\lambda \neq 0$, invariant under four-parameter algebra of invariance, through transformations*

$$\tilde{t} = T(t) \qquad \tilde{x} = \tau^{-1/2}x - \tau^{-1/2}H(t) \qquad u = v(t, x)\tilde{u}(\tilde{t}, \tilde{x})$$

where $T = \frac{1}{2}\int\frac{\mathrm{d}t}{\tau(t)}$, $v(t, x)$ has the form of (31), where $\tau \neq 0$ is any solution of the first equation of (34). $k(\omega)$ is a solution of equation (33) and is reduced to the equation

$$\tilde{u}_{\tilde{t}} = 2k''(\omega)\tilde{u} + 2k'(\omega)\tilde{u}_\omega + \tilde{u}_{\omega\omega}.$$

Remark. By making the following replacement in the last equation:

$$\bar{t} = \tilde{t} \qquad \bar{x} = \omega \qquad \tilde{u} = \exp(k(\omega))\bar{u}$$

and taking account condition (33), this equation is reduced to the following Schrödinger equation:

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \frac{\lambda}{\bar{x}^2}\bar{u}.$$

Thus, in a FPE with a four-parametrical group of symmetry there exists an 'initial' equation to which they are reduced; however, the equation is not a FPE as it takes place in a six-parametrical group.

5. Homogeneous examples of the FPEs having six- and four-parametrical groups of symmetry and some classes of exact solutions

In section 2 we specified a necessary and sufficient condition reducing FPE (1) with $B = 1$ to a homogeneous FPE with $B = 1$, i.e. to the equation with a coefficient of drift $A = A(x)$. The appropriate replacement variables $(t, x, u) \to (\tilde{t}, \tilde{x}, \tilde{u})$ were also constructed. Now we list examples of the homogeneous equations which are frequently met in applications.

5.1. Equations which are reduced to the heat equation

(1) The equation describing the Ornstein–Uhlenbeck process [1]:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial k}(kxu) + \frac{1}{2}D\frac{\partial^2 u}{\partial x^2}. \tag{35}$$

Here $A = -kx$, $B = D =$ const. Furthermore, in all examples where $B =$ const $\neq 0$ it is possible, without restriction of generality, to put $B = 1$. Using the replacement

$$u(t, x) = v(t, x)\tilde{u}(\tilde{t}(t), \tilde{x}(t, x)) \tag{36}$$

where $v(t, x)$, $\tilde{t}$, $\tilde{x}$ are found from formulae (22), (24)–(26):

$$v = \exp(kt) \qquad \tilde{x} = \exp(kt)x \qquad \tilde{t} = \frac{1}{4k}\exp(2kt)$$

where the Ornstein–Uhlenbeck equation (35) results in heat equation (27).

(2) Diffusional process in a field of force, of weight [1]:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(gu) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}. \tag{37}$$

Equation (37) is reduced to (27) through replacement (36), where

$$v = \exp\left(-gx - \frac{g^2}{2}t\right) \qquad \tilde{x} = x \qquad \tilde{t} = \frac{1}{2}t.$$

(3) The equation of a Rayleigh-type process [15]:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left(\gamma x - \frac{1}{x}\right)u\right] + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{38}$$

is reduced to (27) through replacement (36), where

$$v = \exp(2\gamma t)x \qquad \tilde{x} = \exp(\gamma t)x \qquad \tilde{t} = \frac{1}{4\gamma}\exp(2\gamma t).$$

Let us consider three equations (39)–(41) describing models in population genetics [6]:

(4) $$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}[(1 - x^2)^2 u]. \tag{39}$$

Using replacement [10]

$$u = \frac{1}{\sqrt{B(x)}} w(\tau, y) \qquad \tau = t \qquad y = \int \frac{\mathrm{d}x}{\sqrt{B(x)}}$$

equation (39) results in

$$W_\tau = -(A(y)w)\tau + \tfrac{1}{2} w_{yy}$$

where $A(y) = \sqrt{2}th(\sqrt{2}y)$, $B = 1$. It is easy to verify that $A(y)$ satisfies condition (20) ($\lambda = 0$). Then the superposition of the above transformation and the corresponding replacement (22), (24)–(26) gives replacement (36), where

$$v = \exp(-t)(1 - x^2)^{-3/2} \qquad \tilde{x} = \frac{1}{2} \ln \frac{1 + x}{1 - x} \qquad \tilde{t} = t$$

which transforms equation (39) into heat equation (27).

(5)
$$\frac{\partial u}{\partial t} = \frac{\alpha}{2} \frac{\partial^2}{\partial x^2}[x^2(1 - x^2)u]. \tag{40}$$

With the method used in item (4) we get replacement (36), where

$$v = \exp\left(-\frac{\alpha}{8}t\right) x^{-3/2}(1 - x^2)^{-3/2} \qquad \tilde{x} = \ln \frac{x}{1 - x} \qquad \tilde{t} = \frac{\alpha}{2}t.$$

(6)
$$\frac{\partial u}{\partial t} = \frac{\alpha}{2} \frac{\partial^2}{\partial x^2}[(x - c)^2 u] + \beta \frac{\partial}{\partial x}[(x - c)u]. \tag{41}$$

Equation (41) is reduced to heat equation (27) through replacement (36), where

$$v = \exp\left\{-\left(\frac{\beta^2}{2\alpha} + \frac{\beta}{2} + \frac{\alpha}{8}\right) t\right\} (x - c)^{(-3/2+\beta/\alpha)} \qquad \tilde{x} = \sqrt{2/\alpha}\ln(x - c) \qquad \tilde{t} = t.$$

(7) A FPE of the type [9]

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial k}[a(t)x + b(t)]u + c(t)\frac{\partial^2 u}{\partial x^2} \tag{42}$$

is transformed into heat equation (27) by replacement (36), where [15]

$$v = \exp\{\alpha(t)\} \qquad \tilde{x} = \exp\{\alpha(t)\}x + \beta(t) \qquad \tilde{t} = \gamma(t)$$

and, furthermore,

$$\alpha(t) = -\int_0^t a(s)\,\mathrm{d}s \qquad \beta(t) = \int_0^t b(s)\exp\{\alpha(s)\}\,\mathrm{d}s \qquad \gamma = \int_0^t c(s)\exp\{2\alpha(s)\}\,\mathrm{d}s.$$

5.2. Equations with four-parametrical invariance group

(1) In [5] the following equation was considered:

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(\alpha x u) + \frac{\partial^2}{\partial x^2}(\beta x u). \tag{43}$$

Using the replacement

$$u = \frac{1}{\sqrt{2\beta x}} w(\tau, y) \qquad \tau = t \qquad y = \sqrt{2x/\beta}$$

we transform equation (43) into the following FPE:

$$\frac{\partial w}{\partial \tau} = -\frac{\partial}{\partial y}\left[\left(\frac{\alpha y}{2} - \frac{1}{2y}\right) w\right] + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}$$

where $A(y) = (\frac{\alpha y}{2} - \frac{1}{2y})$ satisfies condition (20).

(2) Let us consider a FPE [6]

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{4}x^{-2p}u\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}x^{1-2p}u\right) \tag{44}$$

where $p \neq -\frac{1}{2}$ (the case $p = \frac{1}{2}$ corresponds to a six-parametrical group: see example (6)). Using the replacement

$$u = \sqrt{2}x^{(2p-1)/2}w(\tau, y) \qquad \tau = t \qquad y = \frac{2\sqrt{2}}{2p+1}x^{(2p+1)/2}$$

we find that equation (44) is reduced to the following FPE:

$$\frac{\partial w}{\partial \tau} = -\frac{\partial}{\partial y}\left(\frac{1}{2y}w\right) + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}$$

where $A(y) = \frac{1}{2y}$ satisfies condition (20).

(3) We consider the equation describing the Rayleigh process:

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[\left(\frac{\mu}{x} - \gamma x\right)u\right] + \mu\frac{\partial^2 u}{\partial x^2} \tag{45}$$

where γ, μ are arbitrary constants. The function $\frac{A(x)}{2\mu} = \frac{1}{2x} - \frac{\gamma x}{2\mu}$, satisfies condition (20). The invariance group of equation (45) is also neither isomorphic to the invariance group of the heat equation nor reduced to it with the help of local transformations.

Theorem 5. *The maximal invariance algebra of equation (45) is a four-dimensional algebra A_4 with the basic operators*

$$\begin{aligned} &P_0 = \frac{\partial}{\partial t} \qquad D_1 = \exp\{2\gamma t\}\left(\frac{1}{\gamma}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \left(1 - \frac{\gamma}{\mu}x^2\right)I\right) \\ &I = u\frac{\partial}{\partial u} \qquad D_2 = \exp\{-2\gamma t\}\left(\frac{1}{\gamma}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - I\right) \end{aligned} \tag{46}$$

which satisfy the commutation relations

$$\begin{aligned} &[P_0, D_1] = 2\gamma D \qquad [D_1, D_2] = \frac{4}{\gamma}P_0 + 4I \\ &[P_0, D_1] = 2\gamma D \qquad [I, P_0] = [I_1, D_1] = [I_1, D_2] = 0. \end{aligned} \tag{47}$$

The proof can be obtained by Lie's method. Let us take advantage of operators (46) for construction of the exact solutions of equation (1). First of all, we show that its known stationary solution [1]

$$u_s(x) = \frac{\gamma x}{\mu}\exp\left(-\frac{\gamma x^2}{2\mu}\right) \tag{48}$$

is the special case of the solution invariant under the operator D_1 (46). For this purpose we construct the ansatz (substitution) invariant under D_1. According to the algorithm of [18] if the equation has operator symmetry, then an exact solution can be found in the form

$$u(x) = \psi(x)\varphi(\omega)$$

where $\psi(x)$ is a function defined from the condition

$$\begin{aligned} &Q\psi(x) \equiv [\xi^0(t, x)\partial_t + \xi^1(t, x)\partial_x + \eta(t, x, u)]\psi(x) = 0 \\ &\xi^0(t, x)\partial_t\omega + \xi^1(t, x)\partial_x\omega = 0. \end{aligned}$$

Table 1.

Subalgebra	Invariant variable	Ansatz
I	$\omega_1 = t, \omega_2 = x$	$u = 0$
$\frac{1}{\gamma}P_0 + (1+\alpha)I$	$\omega = x$	$u = \exp\{\gamma(1+\alpha)t\}\varphi(\omega)$
D_2	$\omega = \exp\{\gamma t\}x$	$u = \exp\{\gamma t\}\varphi(\omega)$
$-\frac{1}{2}D_2 \pm I$	$\omega = \exp\{\gamma t\}x$	$u = \exp\{\gamma t \pm \mathrm{e}^{2\gamma t}\}\varphi(\omega)$
$\frac{1}{2}D_1 - \frac{1}{2}D_2 + \alpha I$	$\omega = \frac{x^2\mathrm{e}^{2\gamma t}}{\mathrm{e}^{4\gamma t}+1}$	$u = \exp\left\{\gamma t - \frac{\gamma}{2\mu}\frac{\mathrm{e}^{4\gamma t}x^2}{\mathrm{e}^{4\gamma t}+1} + \alpha \mathrm{arctg}(\mathrm{e}^{2\gamma t})\right\}\varphi(\omega)$

An anzatz leads to the reduction of a FPE to an ordinary differential equation (ODE) for a function $\varphi(\omega)$. For the operator D_1 this ansatz has the form

$$u(t,x) = x\exp\left(-\frac{\gamma x^2}{2\mu}\right)\varphi(\omega) \qquad \omega = x\exp(\gamma t). \tag{49}$$

The substitution of (49) into (45) gives the ODE

$$\omega\ddot{\varphi} + \dot{\varphi} = 0 \tag{50}$$

where the overdot designates differentiation on ω. The general solution of equation (50) is

$$\varphi(\omega) = c_1 \ln\omega + c_2. \tag{51}$$

The substitution of (51) into (49) gives the following solution of equation (45):

$$u(t,x) = x\exp\left\{-\frac{\gamma x^2}{2\mu}\right\}(c_1(\gamma t + \ln x) + c_2). \tag{52}$$

It is easy to see that solution (48) follows from (52) when $c_1 = 0$, $c_2 = \gamma/\mu$. For a more regular description of the solutions of equation (45) we find one-dimensional unequivalent subalgebras of the invariance algebra A_4 (46), (47), and then for every such subalgebra we construct an invariant ansatz. Firstly, we notice that algebra A_4 is isomorphic to algebra $AGL(2,R) = ASL(2,R) \oplus I$; the isomorphism is achieved by the linear transformation

$$B_1 = \frac{1}{\gamma}P_0 + I \qquad B_2 = \tfrac{1}{2}D_1 \qquad B_3 = -\tfrac{1}{2}D_2 \qquad B_4 = I.$$

Using this fact, we construct table 1 showing one-dimensional subalgebras and the appropriate invariant ansatze.

As a result of the substitution of ansatze 2–5 from table 1 in (45) we have following reduced ODEs:

$$\begin{aligned}
&(2) \qquad \mu\omega^2\ddot{\varphi} + (\gamma^3 - \mu\omega)\dot{\varphi} + (\mu - \alpha\omega^2)\varphi = 0\\
&(3) \qquad \omega^2\ddot{\varphi} - \omega\dot{\varphi} + \varphi = 0\\
&(4) \qquad \mu\omega^2\ddot{\varphi} - \mu\omega\dot{\varphi} + (\mu \pm 2\gamma\omega^2)\varphi = 0\\
&(5) \qquad 4\mu^2\omega^2\ddot{\varphi} + (\gamma^2\omega^2 - 2\gamma\alpha\mu\omega + \mu^2)\varphi = 0.
\end{aligned} \tag{53}$$

In particular, it is easy to find the general solution of the third equation of (53) and, using the appropriate third ansatz from the table, to obtain the particular solution of equation (45):

$$u(t,x) = x\exp\{2\gamma t\}(c_1 + c_2(\ln x + \gamma t)).$$

6. Conclusions

We found the criterion when a FPE with a time-dependent drift coefficient can be reduced to a homogeneous FPE with the help of local transformations. It appears that it is connected with symmetry properties of this equation: namely, a FPE is reduced to a homogeneous equation if and only if even one operator of symmetry different from $u\partial_u$ exists. From this criterion and previous results [14] it turns out that the dimension of the group of symmetry of a one-dimensional FPE can equal 1, 2, 4 or 6. Moreover, in the case where a FPE is invariant under six- or four-parametrical groups they can be reduced with the help of local transformations to the 'initial' equations: namely, to the heat equation or to the Schrödinger one with the potential $V(x) = \frac{\lambda}{x^2}$, respectively. It is worth noting that specific FPEs in applications are invariant, as a rule, under groups of symmetry with the given dimensions. If one has the solutions of these two 'initial' equations it is possible to build the solutions of an appropriate FPE. In the case of the Schrödinger equation solutions can be found by the reduction method to the ODEs, using one-dimensional subalgebras of its four-dimensional algebra.

The questions of how to construct a transformation to reduce a FPE with a two-dimensional group of symmetry to a homogeneous FPE and whether or not there is an appropriate 'initial' equation still remain open. However, here, as well as in the case of a one-dimensional group of symmetry, the approach of non-local transformations (for example, Darboux transformations) is possible. It would also be interesting to investigate conditional and Q-conditional invariancy, which generalizes the concept of Lie invariancy (see [18]).

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